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Dynamic Approach for Financial Asset Price by Feynman –KAC Formula

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Introduction:

Diffusion processes (specifically, Brownian motion) originated in physics as mathematical models of the motions of individual molecules undergoing random collisions with other molecules in a gas or fluid. Long before the mathematical foundations of the subject were laid, Albert Einstein realized that the microscopic random motion of molecules was ultimately responsible for the macroscopic physical phenomenon of diffusion, and made the connection between the volatility parameter σ of the random process and the diffusion constant in the partial differential equation governing diffusion. The connection between the differential equations of diffusion and heat flow and the random process of Brownian motion has been a recurring theme in mathematical research ever since. In the 1940s, Richard Feynman discovered that the Schrodinger equation (the differential equation governing the time evolution of quantum states in quantum mechanics) could be solved by (a kind of) averaging over paths, an observation which led him to a far-reaching reformulation of the quantum theory in terms of path integrals. Upon learning of Feynman's ideas, Mark Kac (a mathematician at Cornell University, where Feynman was, at the time, an Assistant Professor of Physics) realized that a similar representation could be given for solutions of the heat equation (and other related diffusion equations) with external cooling terms. This representation is now known as the Feynman-Kac formula. Later it became evident that the expectation occurring in this representation is of the same type that occurs in derivative security pricing.

The simplest heat equation with a cooling term is

$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2} - V(x)\, u \quad \text{.............................. (1)}$$

where V(x) is a function of the space variable x representing the amount of external cooling at location x.

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The Feynman-Kac formula should come as no surprise that the mathematical literature is rich in generalizations and variations. Two types of generalizations are of particular usefulness in financial applications: (1) those in which the Brownian motion Wt is replaced by another diffusion process, and (2) those where the Brownian motion (or more generally diffusion process) is restricted to stay within a certain region of space.

Application of the Feynman-Kac Theorems:

In financial applications, the Feynman-Kac theorems are most useful in problems where the expectation giving the arbitrage price of a contract cannot be evaluated in closed form. One must then resort to numerical approximations. Usually, there are two avenues of approach: (1) simulation; or (2) numerical solution of a PDE (or system of PDEs). The Feynman-Kac theorems provide the PDEs. It is not our business in this course to discuss methods for the solution of PDEs. Nevertheless, we cannot leave the subject of the Feynman-Kac formula without doing at least one substantial example. This example will show how the method of Eigen function expansion works, in one of the simplest cases. The payoff will be an explicit formula for the transition probabilities of Brownian motion restricted to an interval.

Transition probabilities for Brownian motion in an interval:

As we consider one-dimensional Brownian motion with killing or absorption at the endpoints of an interval J.

For simplicity, take J = (0, 2π). Recall that τ = τ J is the time of first exit from J by the process Wt. We are interested in the expectation

U (t ,x) = E f(Wt)1 {t < τ} ,

Where f: J → R is a continuous function with compact support in J. This expectation is an instance of the expectation in equation (1), with K(x) ≡ 0. By Theorem 7, the function u satisfies the heat equation with K = 0. Our objective is to find a solution to this differential equation that also satisfies the initial and boundary conditions. Our strategy is based on the superposition principle. Without the constraints of the initial and boundary conditions, there are infinitely many solutions to the heat equation, as we have already seen (look again at equation **V(t , x) =** $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n!)} \frac{d^n}{dt^n} e^{-1/t^2}$

Because the heat equation is linear, any linear combination of solutions is also a solution. Thus, one may attempt to find a solution that satisfies the infinitely and boundary conditions by looking at super positions of simpler solutions. What are the simplest bounded solutions of the heat equation? Other than the constants, probably the simplest are the exponentials

U (t, x : θ) = exp {iθx} exp{-θ^2t/2}

By themselves, these solutions are of no use, as they are complex-valued, and the function u(t, x) defined by real-valued. However, the functions u(t, x; θ) come naturally in pairs, indexed by ±θ. Adding and subtracting the functions in these pairs leads to another large simple class of solutions:

Theoretical Framework of Feynman –Kac Formula.

The Feynman–Kac formula named after Richard Feynman and Mark Kac, established a link between parabolic partial differential equations (PDEs) and stochastic processes. It offers a method of solving certain PDEs by simulating random paths of a stochastic process. Conversely, an important class of expectations of random processes can be computed by deterministic methods.

$\frac{\partial u}{\partial t}$ (x, t) + $\boldsymbol{\mu}$(x,t) $\frac{\partial u}{\partial x}$ +(x,t) + $\frac{1}{2}\delta^2$ (x,t) $\frac{\partial^2 u}{\partial x^2}$ (x,t) – v(x,t) u(x,t) + f(x,t) = 0 , defined for all *x* in **R** and *t* in [0, *T*], subject to the terminal condition

u(x,t) = Ψ (x) , where μ, σ, ψ, *V*, *f* are known functions, *T* is a parameter and u : R × [0,T] → R is the unknown. Then the Feynman–Kac formula tells us that the solution can be written as a conditional expectation

u(x,t) = E^Q [$\int_t^T e^{-\int_t^r v(X_\tau,\tau)d\tau}$ f (x, r) dr + $e^{-\int_t^T v(X_\tau,\tau)d\tau}$ $\Psi(X_t)$ | X_t = x] under the probability measure Q such that *X* is an Itō process driven by the equation dx = $\boldsymbol{\mu}$(X,t) dt + σ (X , t) dW^Q , with $W^Q(t)$ is a Wiener process (also called Brownian motion) under *Q*, and the initial condition for *X*(*t*) is *X*(t) = *x*.

Proof of Feynman–Kac Formula:

Let *u*(*x*, *t*) be the solution to above PDE. Applying Ito's lemma to the process

$$Y(s) = e^{-\int_t^s v(X_\tau,\tau)d\tau}\, u(X_s, s) + \int_t^s e^{-\int_t^r v(X_\tau,\tau)d\tau}\, f(X_r, r)\, dr$$

We gets

dY = d ($e^{-\int_t^s v(X_\tau,\tau)d\tau}$) u (X_s ,s) + $e^{-\int_t^s v(X_\tau,\tau)d\tau}$ du (X_s ,s) + d ($e^{-\int_t^s v(X_\tau,\tau)d\tau}$) du (X_s ,s) + d ($\int_t^s e^{-\int_t^r v(X_\tau,\tau)d\tau}$ f (X_r, r) dr)

Since, d ($e^{-\int_t^s v(X_\tau,\tau)d\tau}$) = - V(X_s ,s) $e^{-\int_t^s v(X_\tau,\tau)d\tau}$ ds , the third term is O (dt du) and can be dropped. We also have that d ($\int_t^s e^{-\int_t^r v(X_\tau,\tau)d\tau}$ f (X_r, r) dr) = $e^{-\int_t^s v(X_\tau,\tau)d\tau}$ f (X_s ,s) ds .

Applying Ito's lemma once again to du (X_s ,s) , it follows that

dY = $e^{-\int_t^s v(X_\tau,\tau)d\tau}$ (-V(X_s , s) u (X_s , s) + f (X_s , s) + $\boldsymbol{\mu}$ (X_s , s) $\frac{\partial u}{\partial x}$ + $\frac{\partial u}{\partial s}$ + $\frac{1}{2}\delta^2$ (X_s , s) + $\frac{\partial^2 u}{\partial x^2}$) ds + $e^{-\int_t^s v(X_\tau,\tau)d\tau}$ $\boldsymbol{\sigma}$(X , s) $\frac{\partial u}{\partial x}$ dW

The first term contains, in parentheses, the above PDE and is therefore zero. What remains is

dY = $e^{-\int_t^s v(X_\tau,\tau)d\tau}$ $\boldsymbol{\sigma}$(X , s) $\frac{\partial u}{\partial x}$ dW .

Integrating this equation from *t* to *T*, one concludes that

Y(T) – Y(t) = $\int_t^T e^{-\int_t^s v(X_\tau,\tau)d\tau}$ $\boldsymbol{\sigma}$(X , s) $\frac{\partial u}{\partial x}$ dW .

Upon taking expectations, conditioned on $X_t = x$, and observing that the right side is an Ito integral, which has expectation zero, it follows that

E [Y(T) | X_t = x] = E Y(t) | X_t = x] = u (x , t) .

The desired result is obtained by observing that

E [Y(T) | X_t = x] = E [$e^{-\int_t^s v(X_\tau,\tau)d\tau}$ u(X_t , T) $\int_t^T e^{-\int_t^s v(X_\tau,\tau)d\tau}$ (f (X_r, r) dr | X_t = x] and finally

u (x ,t) = E [$e^{-\int_t^T v(X_\tau,\tau)d\tau}$ Ψ(X_T) $\int_t^T e^{-\int_t^s v(X_\tau,\tau)d\tau}$ (f (X_s , s) ds | X_t = x] .

The Feynman–Kac formula can also be interpreted as a method for evaluating functional integrals of a certain form. If

I = ∫ f (x (0)) $e^{-u\int_0^t V(x(t))dt}$ g(x(t)) Dx

where the integral is taken over all random walks, then

I = ∫ w (x , t) g (x) dx

where *w*(*x*, *t*) is a solution to the parabolic partial differential equation

$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial x^2}$ - u V(x)w with initial condition $w(x, 0) = f(x)$.

Methodology:

The partial differential equation that describes the expected final price of an asset whose price is a stochastic process given by a stochastic differential equation.

The following methods that have for our research...................

1. We define Parameters of the Model Using Stochastic Differential Equations
2. Applying Ito's Rule
3. Solve Feynman-Kac Equation
4. Compute Expected Time to Sell Asset

1 : The Parameters of the Model Using Stochastic Differential Equations

The model for the price of an asset X(t) defined in the time interval [0, T] is a stochastic process defined by a stochastic differential equation of the form

$dX = \mu(t, X)dt + \sigma(t, X)dB(t)$, where B(t) is the Wiener process with unit variance parameter.

- r(t) is a continuous function representing a spot interest rate. This rate determines the discount factor for the final payoff at the time T .
- u(t,x) is the expected value of the discounted future price calculated as

$X(T)\exp\left(-\int_t^T r(t)dt\right)$ under the condition X(t) = x.

- $\mu(t, X)$ and $\sigma(t, X)$ are drift and diffusion of the stochastic process X(t).

According to the Feynman-Kac theorem, u satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial u}{\partial x} - ur = 0$$

With a final condition at time T.

To transform the final condition into an initial condition, apply a time reversal by setting

v(t, X)= u(T - t, X).

2 : We apply the Ito's Rule

Asset prices follow a multiplicative process. That is, the logarithm of the price can be described in terms of an SDE, but the expected value of the price itself is of interest because it describes the profit, and thus we need an SDE for the latter.

In general, if a stochastically process X is given in terms of an SDE, then Ito's rule says that the transformed process G(t, X) satisfies

$$dG = \left(\mu \frac{dG}{dX} + \frac{\sigma^2}{2}\frac{d^2G}{d X^2} + \frac{dG}{dt}\right)dt + \frac{dG}{dX}\sigma dB(t)$$

We assume that the logarithm of the price is given by a one-dimensional additive Brownian motion, that is, mu and sigma are constants. Define a function that applies Ito's rule, and use it to transform the additive Brownian motion into a geometric Brownian motion.

For simplicity, we assume that the interest rate is zero. This is a special case also known as Kolmogorov backward equation.

3 : To Solve Feynman-Kac Equation :

As the final condition, take the identity mapping. That is, the payoff at time T is given by the asset price itself. You can modify this line in order to investigate derivative instruments.

Numerical solving of PDEs can only be applied to a finite domain. Therefore, you must specify a boundary condition. Assume that the asset is sold at the moment when its price rises above or falls below a certain limit, and thus the solution v has to satisfy x - v = 0 at the boundary points x. You can choose another boundary condition, for example, you can use v = 0 to model knockout options. The zeroes in the second and fourth output indicate that the boundary condition does not depend on $\frac{\partial v}{\partial x}$.

We choose the space grid, which is the range of values of the price x. Set the left boundary to zero: it is not reachable by a geometric random walk. Set the right boundary to one: when the asset price reaches this limit, the asset is sold and choose the time grid. Because of the time reversal applied in the beginning, it denotes the time left until the

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final time T. The expected selling price depends nearly linearly on the price at time t, and also weakly on t .

The state of a geometric Brownian motion with drift $\mu 1$ at time t is a log normally distributed random variable with expected value exp (μ_1 , t)times its initial value. This describes the expected selling price of an asset that is never sold because of reaching a limit and dividing the solution obtained above by that expected value shows how much profit is lost by selling prematurely at the limit.

The ratio of the expected payoff of an asset for which a limit sales order has been placed and the same asset without sales order over a time span T, as a function of t. Consider the case of an order limit of two and four times the current price, respectively.

4 : We compute Expected Time to Sell Asset :

It is the expected exit time when the limit is reached and the asset is sold is given by the following equation:

In addition, y must be zero at the boundaries. For insert the actual stochastic process under consideration:

To solve this problem for arbitrary interval borders a and b.

L is undefined at a = 0. Set the assumption that 0 < X < 1.

Using the value b = 1 for the right border, compute the limit.

Finding and Analysis:

eq(t, X) = diff(u(t, X), t) + (sigma(t, X)^2*diff(u(t, X), X, X))/2 + mu(t, X)*diff(u(t, X), X) - r(t, X)*u(t, X) .

eq2 = (sigma(T - t, X)^2*diff(v(t, X), X, X))/2 - diff(v(t, X), t) + mu(T - t, X)*diff(v(t, X), X) - v(t, X)*r(T - t, X) .

k = [-1, sigma(T - t, X)^2/2, mu(T - t, X)*diff(v(t, X), X) - v(t, X)*r(T - t, X)]

terms = [dvt, dvXX, 1]

s(t, X) = mu(T - t, X)*diff(v(t, X), X) - v(t, X)*r(T - t, X) - sigma(T - t, X)*D([2], sigma)(T - t, X)*diff(v(t, X), X)

Applying Ito Rule:

mu1 = (exp(X)*sigma0^2)/2 + mu0*exp(X)

sigma1 = sigma0*exp(X)

s = subs(diff(v(t, X), X), X, log(Y))*mu(T - t, log(Y)) - v(t, log(Y))*r(T - t, log(Y)) - D([2], sigma)(T - t, log(Y))*subs(diff(v(t, X), X), X, log(Y))*sigma(T - t, log(Y)) .

f = (Y^2*sigma0^2*subs(diff(v(t, X), X), X, log(Y)))/2 .

Solution to the Feynman-Kac Equation

s0 = (Y*dvdx)/2

c0 = 1

f0 = (Y^2*dvdx)/2

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```
FeynmanKacPde = function_handle with value:
@(t,Y,V,dvdx)deal(1.0,Y.^2.*dvdx.*(1.0./2.0),Y.*dvdx.*(1.0./2.0))
FeynmanKacIC =  function_handle with value:
@(x)x
FeynmanKacBC =  function_handle with value:
 @(xl,vl,xr,vr,t)deal(xl-vl,0,xr-vr,0)
xgrid = Columns 1 through 11
 0   0.0500   0.1000   0.1500   0.2000   0.2500   0.3000   0.3500   0.4000   0.4500   0.5000
Columns 12 through 21
 0.5500   0.6000   0.6500   0.7000   0.7500   0.8000   0.8500   0.9000   0.9500   1.0000
tgrid =
 0   0.1000   0.2000   0.3000   0.4000   0.5000   0.6000   0.7000   0.8000   0.9000   1.0000
sol = Columns 1 through 11
 0   0.0500   0.1000   0.1500   0.2000   0.2500   0.3000   0.3500   0.4000   0.4500   0.5000
0.0151   0.0558   0.1042   0.1536   0.2033   0.2532   0.3031   0.3530   0.4029   0.4529   0.5028
0.0681   0.0854   0.1237   0.1690   0.2167   0.2653   0.3144   0.3638   0.4133   0.4630   0.5127
0.1328   0.1422   0.1675   0.2034   0.2454   0.2906   0.3375   0.3853   0.4337   0.4824   0.5314
0.2097   0.2159   0.2336   0.2606   0.2946   0.3337   0.3760   0.4206   0.4666   0.5135   0.5611
0.2983   0.3029   0.3161   0.3369   0.3641   0.3965   0.4331   0.4728   0.5147   0.5583   0.6030
0.3978   0.4013   0.4116   0.4280   0.4500   0.4767   0.5074   0.5413   0.5777   0.6161   0.6556
0.5056   0.5083   0.5164   0.5293   0.5468   0.5683   0.5932   0.6208   0.6506   0.6821   0.7146
0.6155   0.6175   0.6236   0.6334   0.6466   0.6628   0.6816   0.7027   0.7255   0.7496   0.7747
0.7181   0.7195   0.7238   0.7306   0.7399   0.7513   0.7646   0.7796   0.7959   0.8133   0.8315
0.8048   0.8057   0.8085   0.8131   0.8193   0.8270   0.8360   0.8461   0.8572   0.8691   0.8815
Columns 12 through 21
0.5500   0.6000   0.6500   0.7000   0.7500   0.8000   0.8500   0.9000   0.9500   1.0000
0.5528   0.6028   0.6528   0.7027   0.7527   0.8027   0.8527   0.9027   0.9525   1.0000
0.5624   0.6122   0.6621   0.7119   0.7618   0.8117   0.8615   0.9109   0.9584   1.0000
0.5806   0.6299   0.6794   0.7288   0.7782   0.8273   0.8756   0.9217   0.9638   1.0000
0.6091   0.6573   0.7055   0.7536   0.8009   0.8469   0.8908   0.9315   0.9682   1.0000
0.6483   0.6937   0.7388   0.7830   0.8259   0.8668   0.9051   0.9403   0.9721   1.0000
0.6957   0.7358   0.7754   0.8139   0.8508   0.8859   0.9185   0.9486   0.9758   1.0000
0.7476   0.7807   0.8133   0.8450   0.8755   0.9045   0.9316   0.9567   0.9795   1.0000
0.8002   0.8259   0.8514   0.8763   0.9003   0.9231   0.9447   0.9648   0.9833   1.0000
0.8502   0.8690   0.8878   0.9063   0.9242   0.9413   0.9576   0.9729   0.9871   1.0000
0.8943   0.9074   0.9204   0.9333   0.9459   0.9580   0.9696   0.9805   0.9907   1.0000
```

Computation of Expected Time to Sell Asset:

```
exitTimeEquation(X) = mu(t, X)*diff(y(X), X) + (sigma(t, X)^2*diff(y(X), X, X))/2
== -1

exitTimeGBM(X) = ((X*sigma0^2)/2 + X*mu0)*diff(y(X), X) +
(X^2*sigma0^2*diff(y(X), X, X))/2 == -1 .
L =  -(log(X) - log(a))*(log(X) - log(b)) .
assume(0 < X & X < 1)
limit(subs(L, b, 1), a, 0, 'Right')
 ans = Inf
```

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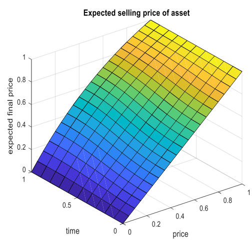
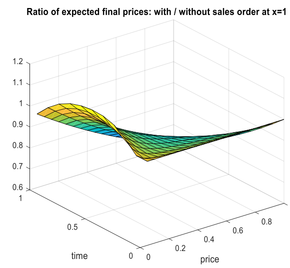
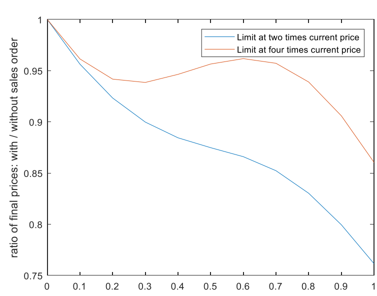
Graphical Representation of Feynman-Kac Equation







Conclusion:

The Feynman–Kac formula, a link between parabolic partial differential equations (PDEs) and stochastic processes. It offers a method of solving certain PDEs by simulating random paths of a stochastic process. Conversely, an important class of

expectations of random processes can be computed by deterministic methods. In this context, we tried to find a numerical solution for the price of an asset X(t) defined in the time interval [0, T] is a stochastic process defined by a stochastic differential equation . The expected payoff of an asset for which a limit sales order has been placed and the same asset without sales order over a time span T, as a function of t. The expected time sell the asset is L = - (log(X) - log (a))*(log(X) - log (b)), where a = 0 and b = 1.
The expected exit time is ∞ (infinite).

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